ON A CLASS OF SOLUTIONS OF GRAD'S MOMENT EQUATIONS

(OB ODNOM KLASSE RESHENII URAVNENII KINETICHESKIKH MOMENTOV GRZDA)

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The values of the stresses and of the heat fluxes in shear flow of an ideal monatomic Maxwellian gas in the absence of external forces have been computed in the papers [1,2] using the moment equations. This made it possible to estimate, through comparison, the area of applicability of the equations of Navier-Stokes, Burnett, etc. and to clarify the effect of rarefaction on the flow parameters. This paper considers a wider class of flows of such a gas under the assumption that all the moments of the distribution function are functions of time t, and the macroscopic velocity depends linearly on the coordinates.

In the absence of external forces the equations of continuity, momentum, energy, the stresses p_{ij} and the third order moments S_{ijk} of an ideal monatomic gas with Maxwellian molecules are, respectively, [3]

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r}(\rho u_r) = 0, \qquad \frac{\partial u_i}{\partial t} + u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial p_{ir}}{\partial x_r} = 0$$

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r}(p u_r) + \frac{2}{3}(p_{ij} + p\delta_{ij})\frac{\partial u_i}{\partial x_j} + \frac{1}{3}\frac{\partial s_r}{\partial x_r} = 0 \qquad (1)$$

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}(u_r p_{ij}) + \frac{\partial S_{ijr}}{\partial x_r} - \frac{1}{3}\delta_{ij}\frac{\partial S_r}{\partial x_r} + 2\left[p_{ir}\frac{\partial u_j}{\partial x_r}\right] + 2p\left[\frac{\partial u_i}{\partial x_j}\right] + \alpha p_{ij} = 0 \qquad (2)$$

$$\frac{\partial S_{ijk}}{\partial t} + \frac{\partial}{\partial x_r}(u_r S_{ijk} + Q_{ijkr}) + S_{ijr}\frac{\partial u_k}{\partial x_r} + S_{irk}\frac{\partial u_j}{\partial x_r} + S_{rjk}\frac{\partial u_i}{\partial x_r} -$$

$$-\frac{1}{\rho}\left(\tau_{ij}\frac{\partial \tau_{kr}}{\partial x_r} + \tau_{ik}\frac{\partial \tau_{jr}}{\partial x_r} + \tau_{jk}\frac{\partial \tau_{ir}}{\partial x_r}\right) + \frac{\alpha}{6}(9S_{ijk} - S_i\delta_{jk} - S_j\delta_{ik} - S_k\delta_{ij}) = 0 \qquad (3)$$

Here we made use of the usual convention of summing over repeated indices, and have put

$$\alpha = \frac{R\rho}{\mu_0}, \qquad \mu_0 = \frac{\mu}{T}, \qquad R = \frac{p}{\rho T}, \text{ heat flow}$$

$$\frac{1}{2} S_r = \frac{1}{2} S_{rkk}, \qquad \tau_{ij} = p_{ij} + p\delta_{ij}, \qquad p_{11} + p_{22} + p_{33} = 0$$

Q_{ijkr} is the fourth order moment, δ_{ij} the unit tensor,

$$[A_{ij}] = \frac{1}{2}(A_{ij} + A_{ji}) - \frac{1}{3} A_{kk}\delta_{ij}$$

Let

$$\rho = \rho(t), \qquad \tau_{ij} = \tau_{ij}(t), \qquad S_{ijk} = S_{ijk}(t)$$

$$Q_{ijkr} = Q_{ijkr}(t), \qquad u_i = \Psi_{ij}(t)\, x_j + \varphi(t)$$

In what follows we shall assume, for simplicity, that $\phi(t) = 0$. Then the equation of continuity implies

$$\rho = \rho_0 \exp\left(-\int I dt\right) \qquad (I = \Psi_{11} + \Psi_{22} + \Psi_{33})$$

and the momentum equation implies

$$\frac{d\Psi_{i1}}{dt} + \Psi_{r1}\Psi_{ir} = 0, \quad \frac{d\Psi_{i2}}{dt} + \Psi_{r2}\Psi_{ir} = 0, \quad \frac{d\Psi_{i3}}{dt} + \Psi_{r3}\Psi_{ir} = 0 \tag{4}$$

Equations (1) - (3) take the form

$$\frac{dp}{dt} + \frac{5}{3} pI + \frac{2}{3} p_{ij}\Psi_{ij} = 0 \qquad \left(\alpha_0 = \frac{R\rho_0}{\mu_0}\right) \tag{5}$$

$$\frac{dp_{ij}}{dt} + p_{ij}I + 2[p_{ir}\Psi_{jr}] + 2p[\Psi_{ij}] + \alpha_0 \exp\left(-\int I dt\right) p_{ij} = 0 \tag{6}$$

$$\frac{dS_{ijk}}{dt} + S_{ijr}\Psi_{kr} + S_{irk}\Psi_{jr} + S_{rjk}\Psi_{ir} + S_{ijk}I +$$

$$+ \frac{1}{6}\alpha_0 \exp\left(-\int I dt\right)(9S_{ijk} - S_i\delta_{jk} - S_j\delta_{ik} - S_k\delta_{ij}) = 0 \tag{7}$$

Thus the problem of determining the second order moments τ_{ij} and the third order moments S_{ijk} is reduced to that of solving two independent systems of 6 and 10 homogeneous linear differential equations of the first order with variable coefficients, the systems being connected by equations (4).

In Burnett's approximation [4] this problem has the following solution:

$$S_r = 0$$

$$\frac{p_{ij}}{p} = -\frac{2}{\alpha}[\Psi_{ij}] + \frac{10}{3\alpha^2}\Psi_{kk}[\Psi_{ij}] - \frac{2}{\alpha^2}[\Psi_{ki}\Psi_{jk}] - \frac{4}{\alpha^2}[[\Psi_{ik}]\Psi_{kj}] + \frac{8}{\alpha^2}[[\Psi_{ik}][\Psi_{kj}]]$$

where the Navier-Stokes approximation is given by the first term

$$p = p(0)\exp\left[-\int_0^t \left(5I + 2\Psi_{ij}\frac{p_{ij}}{p}\right)\frac{dt}{3}\right]$$

Here the ψ_{ij} are the solutions of the system of equations (4).

In the case under consideration the system of Grad's 13-moment equatic [3] yields exact values for the stresses and approximate values for the heat fluxes, which satisfy the system of equation

$$\frac{dS_i}{dt} + \frac{7}{5}S_r\Psi_{ir} + \frac{2}{5}S_r\Psi_{ri} + \frac{7}{5}S_iI + \frac{2}{3}\alpha S_i = 0 \qquad (8)$$

which is valid for small gradients of the flow parameters.

We now consider some very simple flows of the class flows under investigation.

(a) Let $\psi_{ij} = \psi(t)\delta_{ij}$, where δ_{ij} is the Kronecker delta. Then

$$\Psi(t) = \frac{1}{t+c}, \qquad \mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \frac{\mathbf{r}}{t+c}$$

i.e., we have a flow with spherical propagation. The energy equation takes the form

$$\frac{dp}{dt} + \frac{5p}{t+c} = 0 \qquad \text{or} \qquad p = E(t+c)^{-5}$$

It is then easy to obtain

$$p_{ij} = p_{ij}(0)\exp\left[-5\ln(t+c) + \frac{1}{2}\alpha_0(t+c)^{-2}\right]$$

$$S_i = S_i(0)\exp\left[-6\ln(t+c) + \frac{1}{3}\alpha_0(t+c)^{-2}\right]$$

The system of Grad's 13-moment equations gives exact values of all gas-dynamical parameters of the flow. In the Navier-Stokes and Burnett approximations $p_{ij} = 0$, $S_i = 0$.

(b) Let $\psi_{11} = \psi_{22} = \psi_{33} = 0$, i.e., let the density be constant. The system (4) implies that only three of the coefficients ψ_{ij} are not zero. These are connected by the relation

$$\Psi_{nm} = -\Psi_{nl}\Psi_{lm}t + c_{nm}$$

where ψ_{nl}, ψ_{lm}, c_{nm} are constants. For instance

$$u = (-\Psi_{13}\Psi_{32}t + c_{12})y + \Psi_{13}z, \qquad v = 0, \ w = \Psi_{32}y$$

i.e., in the planes y = const. the flow results from superposition of the uniform flow $w = \psi_{32}y$ on the shear flow

$$u = (-\Psi_{13}\Psi_{32}t + c_{12})y + \Psi_{13}z$$

(c) Let $\psi_{ij} = \psi(t)$. Then

$$\Psi = \frac{1}{t+c}, \qquad u = v = w = \frac{x+y+z}{t+c}$$

The velocity is constant on, and perpendicular to, the plane $x + y + z$ = const., i.e. we have an instance of a uniform flow which decreases with time.

(d) Let all the coefficients ψ_{ij} be constant and different from zero; the density is constant ($I = 0$). Then four of the ψ_{ij} are arbitrary:

$$u = -\left(\Psi_{22} + \frac{\Psi_{32}\Psi_{23}}{\Psi_{22}}\right)\xi, \qquad v = \frac{\Psi_{31}\Psi_{22}}{\Psi_{32}}\xi$$

$$w = \Psi_{31}\xi, \qquad \xi = x + \frac{\Psi_{32}}{\Psi_{31}}y + \frac{\Psi_{32}\Psi_{23}}{\Psi_{31}\Psi_{22}}z$$

The velocity of the flow is constant on the stream surface ξ = const., i.e. the flow is a shearing flow [1,2]. In the Navier-Stokes approximation this flow represents a Couette flow in which the temperature of the walls varies with time.

(e) It is easy to see that in the plane case (equation 9, below) we also have a shearing flow if $\psi_{11} = -\psi_{22}$ (ρ= constant). Then ψ_{12}, ψ_{21}, $\psi_{11} = \pm\sqrt{-\psi_{12}\psi_{21}}$ are constant. An interesting feature of this flow is the fact that for a large range of variation

$$|\beta| \gtrsim 1 \qquad \left(\beta = \sqrt{\frac{2}{3}}\,\frac{\Psi_{12} - \Psi_{21}}{\alpha} = 0\,(M\,l\,/\,L)\right)$$

(where M is the Mach number and l/L is the ratio of the mean free path of the molecules to the characteristic dimension of the frow). The real root $\lambda_1 = ar_1$ of the characteristic equation $\lambda^3 + 2a\lambda^2 + a^2\lambda - a^3\beta^2 = 0$ of the system of equations (5) and (6) is better approximated by a series in large β, for which $r_1 = \beta^{2/3} - 2/3 + 1/9\,\beta^{-2/3}$ than by a series in small β, for which $r_1 = \beta^2 - 2\beta^4$.

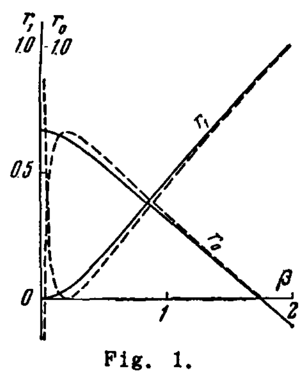

Fig. 1.

A similar situation obtains for the real root $\lambda_0 = ar_0$ of the characteristic equation

$$\lambda^3 + \frac{11}{3}\alpha\lambda^2 + \frac{17}{4}\alpha^2\lambda + \frac{1}{2}(3 - \beta^2)\alpha^3 = 0$$

of the system of equations for the determination of S_3.

In Fig. 1 the exact values of the roots r_1, r_0 (the solid lines) are compared with the approximate values

$$r_1 = \beta^{2/3} - \frac{2}{3} + \frac{1}{9}\beta^{-2/3}, \qquad r_0 = \left(\frac{1}{2}\beta^2\right)^{1/3} - \frac{11}{9} + \frac{25}{324}\left(\frac{1}{2}\beta^2\right)^{-1/3}$$

The system of equation (8) yields $r_0 = -2/3$.

(f) In the plane case the system (4) has a simple solution. In this case

$$\frac{d\Psi_{11}}{dt} + \Psi_{11}^2 + \Psi_{21}\Psi_{12} = 0, \qquad \frac{d\Psi_{21}}{dt} + \Psi_{11}\Psi_{21} + \Psi_{21}\Psi_{22} = 0$$

$$\frac{d\Psi_{12}}{dt} + \Psi_{11}\Psi_{12} + \Psi_{12}\Psi_{22} = 0, \qquad \frac{d\Psi_{22}}{dt} + \Psi_{12}\Psi_{21} + \Psi_{22}^2 = 0 \qquad (9)$$

whence

$$\Psi_{21} = c_1\Psi_{12} = c_1\Psi, \qquad \Psi_{11} - \Psi_{22} = c_2\Psi, \qquad 2\Psi_{11} = -\frac{d\ln\Psi}{dt} + c_2\Psi$$

where ψ satisfies the equation

$$\Psi\frac{d^2\Psi}{dt^2} - \frac{3}{2}\left(\frac{d\Psi}{dt}\right)^2 - \Psi^4\left(\frac{c_2^2}{2} + 2c_1\right) = 0$$

whose solution is

$$\Psi^{-1} = c_3(t - c_4)^2 - c_2^2 - 4c_1$$

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